Derivation of Niclas Carlsson's formula

Let the function *f(x)* be *sin(x)*. We want to evaluate, approximatively, the value of the *n*-th iterate of *f(x)*.

Niclas Carlsson obtained empirically the following formula

$$f^{n}(x) \approx \sqrt{\frac{3}{n}}, x \approx 1, n \text{ large}$$

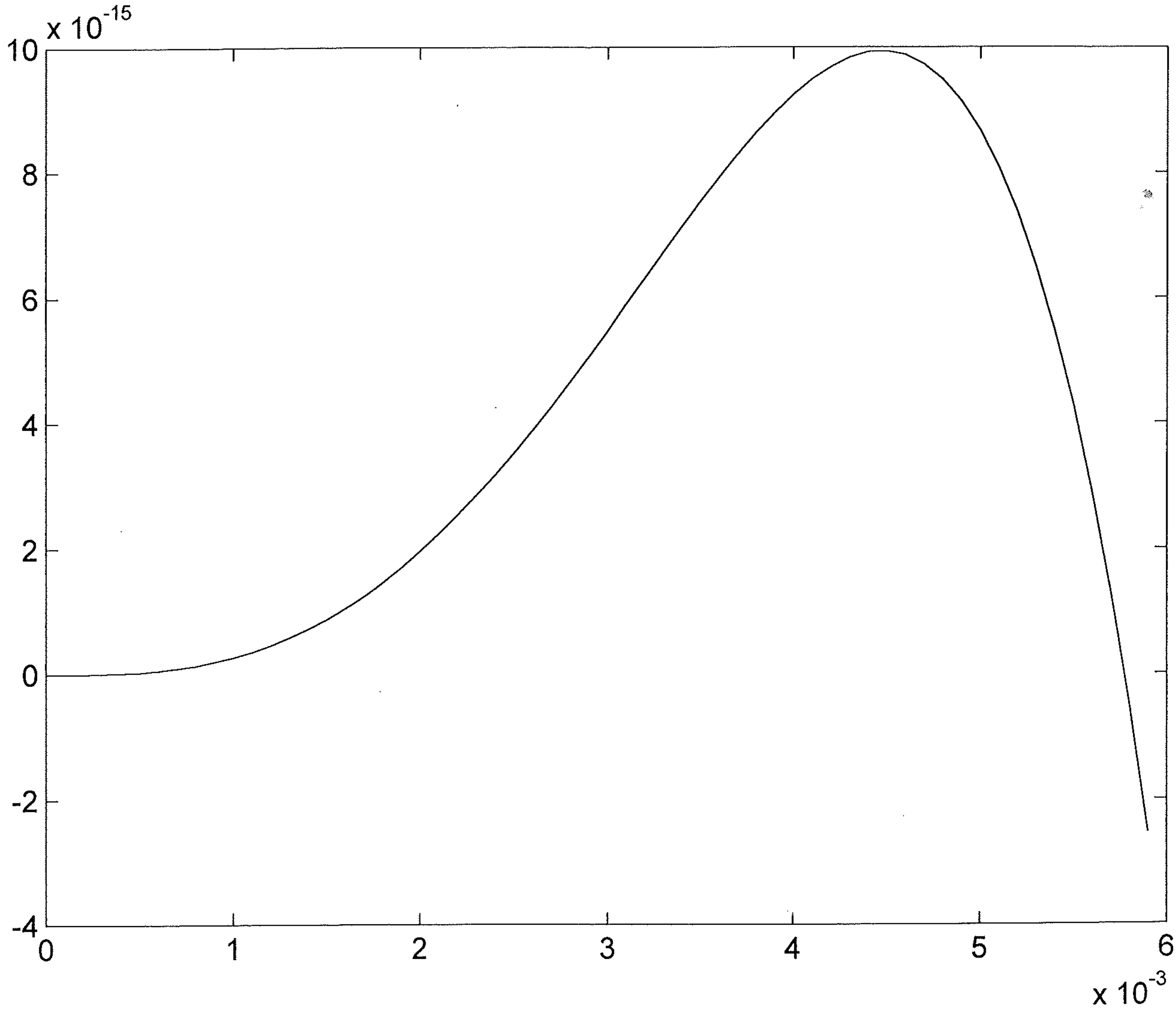
If the formula is correct it will take $3 \cdot 10^{10}$ (30 000 000 000) steps to reach 0.00001 starting from 1.

```
x = 0:.0001:1;

y= sin(x);

low = x - x.^3/6;

hgh = x-x.^3/6.00001;
```

Let us first check whether the approximations are valid. The polynomial function *low* is a lower bound because it is the Maclaurin polynomial of degree 3 of the *sin* function. Since the Maclaurin series is alternating *low* is really less than the *sin* function for positive values of *x*.

We also know that *hgh* is an upper bound for the *sin* function for positive *x* small enough. What is small enough?

```
d = hgh - y;

plot(x(1:60),d(1:60))
```



$$d(x) = x - \frac{x^3}{6{,}00001} - \sin x$$

We see that *hgh* dominates *sin* up on (0, 0.005). The derivation below will show that the number of iterations to reach 0.005 is negligible in comparison with how long it takes to go from 0.005 to 0.00001.

All our functions hgh, f and low are strictly increasing for x in (0,1). Then the following inequality holds:

$$hgh^n(x) > f^n(x) > low^n(x), n = 1,2,...$$

the left inequality on (0, 0.005) and the right one on (0,1).

We will make the derivation for the lower approximation only. By analogous reasoning the upper one gives practically the same result.

Let us first investigate how long it takes for the dynamical system defined by low to descend from a level b to rb where r is a number in (0,1). (We want to think of r as being very close to 1). The step lengths are about $\frac{b^3}{6}$ each so the number of steps required is $\frac{6b(1-r)}{b^3}$. To get further down from rb to r^2b requires another $\frac{6(1-r)}{b^2r^2}$ steps. The total amount of steps needed down to $r^n b$ is thus

$$\frac{6(1-r)}{b^2}\cdot(1+\frac{1}{r^2}+\frac{1}{r^4}+\cdots+\frac{1}{r^{2(n-1)}}) = \frac{6(1-r)}{b^2}\cdot(\frac{r^{-2n}-1}{r^{-2}-1}) = \frac{6r^2}{b^2}\cdot\frac{r^{-2n}-1}{1+r}.$$

Putting $b = 1$, $r^n = 10^{-5}$ and finally taking the limit as $r \to 1$ we get the following estimate for the number of steps: $3\cdot 10^{10}$.

What about the approximate formula by Niclas Carlsson?

To get down to some x from 1 we require, by the above analysis, about $3x^{-2}$ steps. Thus after n steps we have reached the level $\sqrt{\frac{3}{n}}$, i. e.,

$$f^n(x) \approx \sqrt{\frac{3}{n}}.$$

How good is the formula?

Let us iterate the sin function to see how far we get in n steps.

```
x(1)=1;   x(30001)=0;

for i=1:30000, x(i+1)=sin(x(i));end;

nn=1:30001;

p=sqrt(3*nn.^(-1));
```

In the evaluations below the vector has components n, fn and pn. n is the iteration number, fn is the actual value and pn is the value predicted by the formula. The difference is denoted by dn.

```
for i=1:100, n(i)=nn(300*i);fn(i)=x(n(i));pn(i)=p(n(i));end;
```

n	fn	pn	dn
300	0.0992	0.1000	-8.0055e-004
3000	0.0316	0.0316	-3.2813e-005
6000	0.0223	0.0224	-1.2383e-005
15000	0.0141	0.0141	-3.3930e-006
30000	0.0100	0.0100	-1.2691e-006

ans =

After 60000 iterations the value is 0.0071 and the discrepancy -4.7323e-007.

$$x_{n+1} = \sin(x_n)$$

$$x_0 = 1$$

Approximate formula

$$x_n = \sqrt{\frac{3}{n}}$$

n	x_n (MATLAB)	$\sqrt{\frac{3}{n}}$
1000	0.0546	0.0548
10000	0.01731	0.01732
50000	0.007745	0.007746
100 000	0.0054770	0.0054772
500 000	0.002449467	0.002449490
1000000	0.001732042	0.001732051
3000 000	0.000999981035	0.001
12000 000	0.0004999997456	0.0005
51000 000	0.0002425355963	0.00024253562
100 000 000	$1.73205069080 4 \cdot 10^{-4}$	$1.732050 8075 \cdot 10^{-4}$
300 000 000	$10^{-4} - 2.3570082 \cdot 10^{-12}$	10^{-4}